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The Lyapunov Theorem on the Behavior of Solutions of a Nonautonomous Differential System in a Neighborhood of a Periodic Solution with Corresponding Multivalued Variational Equation

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1. INTRODUCTION

The following result, which goes back to Lyapunov, is well known [12, p. 285].

LYAPUNOV'S THEOREM. *Suppose that:*

(i) $f: R^+ \times E^n \to E^n$, $R^+ = [0, \infty)$, *is continuously differentiable, periodic in* t *with period* $p > 0$, *and the equation*

$$\dot{x} = f(t, x) \tag{1}$$

has a periodic solution $y(t) = y(t, t^0, y^0)$, $(t^0, y^0) \in [0, p] \times E^n$, *also of period* p;

(ii) *all the characteristic numbers of the variational equation*

$$\dot{x} = f_x(t, y(t))x \tag{2}$$

have moduli strictly less than 1.

Then there exist positive constants δ, L, c *such that every solution* $x(t) = x(t, t^0, x^0)$ *of* (1) *with* $t^0 \in [0, p]$, $|x^0 - y^0| < \delta$ *satisfies*

$$|x(t) - y(t)| \leqslant L\,|x^0 - y^0| \exp[-c(t - t^0)], \qquad t \in [t^0, \infty). \tag{3}$$

The aim of this paper is to investigate the analog of Lyapunov's theorem when, along the periodic solution $y(t)$ of (1), f may fail of being Fréchet differentiable and (instead of f_x) f possesses a multivalued differential D_f. Then the variational equation (2) is replaced by

$$\dot{x} \in D_f(t, y(t); x).$$

Our main result states, roughly speaking, that if all solutions of the above multivalued variational equation tend to the origin as $t \to \infty$, then every solution $x(t)$ of (1) which starts sufficiently near to $y(t)$, tend to $y(t)$ as $t \to \infty$ in the sense of (3), that is the periodic solution $y(t)$ is asymptotically stable for (1).

In [11] Lasota and Strauss have studied the particular case of the asymptotic stability of the origin for an autonomous equation

$$\dot{x} = f(x)$$

in which f has, at the rest point $x = 0$, a multivalued differential $D_f(0; x)$. Our technique is patterned after that of Lasota and Strauss, though several modifications are incorporated because of the more general nature of the problem that we consider. The definition of multivalued differential which we use is just that introduced in [11], which seems to be a quite natural and useful generalization of Fréchet's differential f_x. While f_x is linear, D_f is merely homogeneous; yet this is apparently enough to retain many of the advantages of the Fréchet differential. Other definitions and applications of multivalued differentials can be found in Banks and Jacobs [1], and in [3, 4, 5, 6].

2. Notation and Preliminaries

Denote by : N the natural numbers, $R^+ = [0, \infty)$, E^n the real Euclidean space with norm $|\cdot|$, $B(x, r)$ the ball $\{y \in E^n : |y - x| < r\}$, $x \in E^n$, $r > 0$, C^n the set of all nonvoid compact convex subsets of E^n endowed with the Hausdorff distance d. Usually we shall write $\|A\|$ in place of $d(A, 0)$, $A \in C^n$. Let Φ be the set of all functions $F\colon A \subseteq R^+ \times E^n \to C^n$ which are uppersemicontinuous (u.s.c.) at each point of A. We recall that a map $F\colon A \to C^n$ is said to be u.s.c. at $p \in A$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $F(q) \subseteq F(p) + B(0, \epsilon)$ whenever $q \in B(p, \delta)$. We denote by χ the set of all $F \in \Phi$ which are homogeneous in the second variable, i.e., satisfy $F(t, sx) = sF(t, x)$, $(t, x) \in R^+ \times E^n$, $s \geqslant 0$.

Let $A \subseteq R^+ \times E^n$.

DEFINITION 1. A continuous function $f: A \to E^n$ is said to be Lipschitzian at $(t, x) \in A$ if there exist constants $L(t)$ and $\delta(t, x)$ such that

$$|f(t, y) - f(t, x)| \leqslant L(t) \,|\, y - x \,|, \quad \text{for all } y \in B(x, \delta(t, x)).$$

If f is Lipschitzian at each point $(t, x) \in A$ and $L(t)$ is continuous f is said to be Lipschitzian in A.

DEFINITION 2. Let $f: A \to E^n$ be Lipschitzian at $(t, x) \in A$. A map $z \to F(t, x; z)$ from E^n to C^n which is u.s.c. and homogeneous in z is called an upper differential of f (with respect to x) at (t, x) if there exists $\delta(t, x) > 0$ such that

$$f(t, y) \in f(t, x) + F(t, x; y - x), \quad \text{for all } y \in B(x, \delta(t, x)).$$

Observe that for such f there always exists an upper differential, namely $F(t, x; y - x) = \bar{B}(0, L(t) \,|\, y - x \,|)$, where – denotes closure.

DEFINITION 3. Let $f: A \to E^n$ be Lipschitzian at $(t, x) \in A$. Define $D_f(t, x; y - x) = \bigcap \{F(t, x; y - x) : F$ is an upper differential of f at $(t, x)\}$. D_f is called multivalued differential of f (with respect to x) at (t, x).

The following lemma, which is essentially due to Lasota and Strauss [11], shows that D_f is well-behaved.

LEMMA 1. *Let $f: A \to E^n$ be Lipschitzian at $(t, x) \in A$. Then $D_f(t, x; z) \in C^n$ for each $z \in E^n$ and, as a function of z, D_f is homogeneous and u.s.c.*

Proof. First we show that $D_f(t, x; z) \neq \varnothing$ for every $z \in E^n$. Let $z = 0$. From $f(t, x) \in f(t, x) + F(t, x; 0)$ we get $0 \in F(t, x; 0)$ for all upper differentials F and $D_f(t, x; 0) \neq \varnothing$. Let $z \neq 0$. Since f is Lipschitzian at (t, x) we have for sufficiently small $s > 0$,

$$\frac{|f(t, x + sz) - f(t, x)|}{s \,|\, z \,|} \leqslant L(t) < \infty.$$

Therefore there exist a point v and a sequence $\{s_k\}$, $s_k > 0$, $s_k \to 0$ such that

$$\frac{f(t, x + s_k z) - f(t, x)}{s_k \,|\, z \,|} \to v \qquad \text{as} \quad k \to \infty.$$

Let F be any upper differential of f at (t, x). Then, if k is large enough, we have

$$\frac{f(t, x + s_k z) - f(t, x)}{s_k \,|\, z \,|} \in \frac{F(t, x; s_k z)}{s_k \,|\, z \,|} = F(t, x; z/|\, z \,|).$$

Let $k \to \infty$. We have $v \in F(t, x; z/|z|)$ from which $|z| v \in F(t, x; z)$. Since F is arbitrary $|z| v \in D_f(t, x; z)$ and this set is nonvoid. It follows easily from the definition that $D_f(t, x; z)$ is homogeneous in z and, for each $z \in E^n$, it is an element of C^n. For the proof of the last statement see [11].

It is shown in [11] that D_f generalizes the notion of Fréchet differential.

EXAMPLE 1. The function $f(t, x) = tx \sin 1/x$, $x \neq 0$, $f(t, 0) = 0$, $t \in R^+$, is Lipschitzian in $R^+ \times E^1$. Set $A = \{(t, 0): t \in R^+\}$. Then

$$D_f(t, x; z) \begin{cases} = f_x(t, x)z, & (t, x) \notin A \\ = t[-|z|, |z|], & (t, x) \in A. \end{cases}$$

This example shows that, for continuous f, one should not expect the multivalued differential D_f to enjoy any reasonable property of continuity as set-valued operator $(t, x) \to D_f(t, x; \cdot)$. However, it appears that in the set A, in which f presents all its differentiability pathology, $D_f(t, 0; z)$ as a set-valued function of (t, z) is certainly continuous in the Hausdorff metric.

DEFINITION 4. A solution of the multivalued differential equation

$$\dot{x} \in F(t, x), \qquad \text{where} \quad F \in \Phi, \tag{4}$$

is defined to be an absolutely continuous function $x(t, t^0, x^0)$, defined on a nondegenerate interval containing t^0, satisfying (4) almost everywhere.

For the theory of Eq. (4) see [7]–[10]. Other applications of multivalued differential equations can be found, for example, in Bushaw [2] and Roxin [13, 14].

3. AUXILIARY THEOREMS

In this paragraph we establish several preparatory theorems, which play a basic role in the proof of our main result. However, some of them can be of interest in themselves. We start with a theorem which is analogous to one proved in [11] by Lasota and Strauss. To prove it we shall use the following

LEMMA 2. *Let* $F: A \to C^n$, $A = [0, T] \times E^n$, $T > 0$, $F \in \Phi$, *satisfy* $\|F(t, x)\| \leqslant a|x| + b$, $(a, b > 0)$, $(t, x) \in A$. *Then, for every* $(t^0, x^0) \in A$ *the multivalued equation* (4) *has at least one solution* $x(t) = x(t, t^0, x^0)$ *defined on some nondegenerate subinterval of* $[0, T]$ *containing* t^0. *Furthermore every solution* $x(t)$ *of* (4) *can be continued to all of* $[0, T]$ *as solution of* (4) *and satisfies for each* $t \in [0, T]$ *the inequality*

$$|x(t)| < 2(|x^0| + b/a) \cosh a(t - t^0). \tag{5}$$

Proof. The first statement follows from the hypothesis that F is u.s.c. (see [9]). Let $[t^0, S)$ be the maximal right interval of existence for $x(t) = x(t, t^0, x^0)$. We want to show that $S = T$. Suppose $S < T$. For almost all $t \in [t^0, S)$ we have, from (4),

$$\left| \frac{d}{dt} \mid x(t) \mid \right| \leqslant a \mid x(t) \mid + b$$

which easily furnishes (5) for all $t \in [t^0, S)$. Thus

$$\mid x(t) \mid < c \qquad \text{where} \quad c = 2(\mid x^0 \mid + b/a) \exp aT, \qquad t \in [t^0, S).$$

But $\lim_{t \to S-} x(t)$ does exist. For, in the contrary case, there exist sequences $\{s_k\}$, $\{t_k\}$, $s_k < t_k$, both converging from the left to S, such that

$$\lim_{k \to \infty} x(s_k) = \bar{x}, \qquad \lim_{k \to \infty} x(t_k) = \bar{\bar{x}}, \qquad \text{where} \quad \bar{x} \neq \bar{\bar{x}}.$$

Then, from

$$x(t_k) - x(s_k) = \int_{s_k}^{t_k} \dot{x}(s)\, ds \in \int_{s_k}^{t_k} F(s, x(s))\, ds$$

we get

$$\mid x(t_k) - x(s_k) \mid \leqslant (ac + b) \mid t_k - s_k \mid$$

and, letting $k \to \infty$, $0 < \mid \bar{x} - \bar{\bar{x}} \mid \leqslant 0$, which is impossible. Thus $\lim_{t \to S-} x(t)$ exists and $x(t)$ can be continued as solution of (4) beyond S, a contradiction. Therefore $S = T$ and, since $\lim_{t \to T-} x(t)$ exists, $x(t)$ is defined all over $[t^0, T]$. In the same way one proves that the left maximal interval of existence of $x(t)$ is $[0, t^0]$. Thus (5) holds for each $t \in [0, T]$ and the proof of the lemma is complete.

THEOREM 1. *Suppose that:*

(i) *$\{F_k\}$ is an infinite sequence of functions $F_k : A \to C^n$, $A = R^+ \times E^n$, $F_k \in \Phi$, such that $F_k(t + p, x) = F_k(t, x)$, $p > 0$, $\| F_k(t, x) \| \leqslant a \mid x \mid + b$, $(a, b > 0)$ and, $F_1(t, x) \supseteq F_2(t, x) \supseteq \ldots$, $(t, x) \in A$;*

(ii) *every solution $x(t)$ of*

$$\dot{x} \in F(t, x) \qquad \textit{where} \quad F(t, x) = \bigcap_{k=1}^{\infty} F_k(t, x), \tag{6}$$

with initial data $(t^0, x^0) \in Q = [0, p] \times \bar{B}(0, r)$, $r > 0$, approaches zero as $t \to \infty$.

Then there exist $k \in N$ and $L \geqslant r$ such that every solution $x(t)$ of

$$\dot{x} \in F_k(t, x), \tag{7_k}$$

with initial values $(t^0, x^0) \in Q$, satisfies $|x(t)| \leqslant L$ for all $t \in R^+$.

Proof. First we observe that, by Lemma 2 and the periodicity of F and F_k, it follows that each solution of (6), (7_k) with initial values in Q can be continued all over R^+ and satisfies (5) for every $t \in R^+$. In particular all solutions $x(t) = x(t, t^0, x^0)$ of (7_k), $k \in N$, with $(t^0, x^0) \in Q$ satisfy

$$|x(t)| < (r + b/a)(\exp at + \exp ap), \qquad \text{for all} \quad t \in R^+. \tag{8}$$

Now suppose the statement of the theorem to be false. Then, for every $k \in N$ there exist $(t^0, x^0) \in Q$ and a solution $x(t) = x(t, t^0, x^0)$ of (7_k) such that

$$|x(t)| > k + H \qquad \text{for some} \quad t \in R^+,$$

where $H = 2(r + b/a) \exp 2ap$. Let $k \in N$. Choose $x_k(t)$ to be any solution of (7_k) with data $(t_k{}^0, x_k{}^0) \in Q$ satisfying the following conditions:

(α) $|x_k(t)| < k + H$, for every $t \in [0, s_k)$, $x(s_k) = k + H$;

(β) each other solution $x(t)$ of (7_k) with initial values $(t^0, x^0) \in Q$ is such that $|x(t)| < k + H$ for all $t \in [t^0, s)$ where $s \geqslant s_k$.

Because of the choice of H we have $s_1 > 2p$. Furthermore, taking into consideration the estimate (8), we deduce that $\limsup_{k\to\infty} s_k = \infty$. By the construction of $x_k(t)$ we have $s_k < s_{k+1}$, $k \in N$. Therefore,

$$2p < s_1 < s_2 \dots, \qquad \lim_{k\to\infty} s_k = \infty.$$

We claim that $x_k(t)$ satisfies

$$|x_k(t)| > r, \qquad \text{for all} \quad t \in [2p, s_k). \tag{9}$$

Suppose the contrary. Then there exists $\bar{t} \in [2p, s_k)$ such that $|x_k(\bar{t})| \leqslant r$ and, for some integer $m \geqslant 3$,

$$(m - 1)p \leqslant \bar{t} < mp.$$

Define

$$z(t) = z(t, \bar{t} - (m - 1)p, \bar{x}) = x_k(t + (m - 1)p) \qquad \text{where} \quad \bar{x} = x_k(\bar{t}).$$

The function $z(t)$ is solution of (7_k), for

$$\dot{x}_k(t + (m - 1)p) \in F_k(t + (m - 1)p, x_k(t + (m - 1)p))$$

and, being F_k periodic in t with period p,

$$\dot{x}_k(t + (m - 1)p) \in F_k(t, x_k(t + (m - 1)p)),$$

that is $\dot{z}(t) \in F_k(t, z(t))$. But $z(t)$ is a solution of (7_k) with data $(\bar{t} - (m - 1)p, \bar{x})$ in Q and reaches the boundary of $\bar{B}(0, k + H)$ at time $s_k - (m - 1)p$ which is strictly less than s_k, a contradiction to (β). Thus the claim (9) is true.

Consider the sequence $\{x_k(t)\}$. As a consequence of inequality (8), $\{x_k(t)\}$ is equibounded on $[p, s_i]$, $i \in N$ fixed, that is we have

$$| x_k(t)| \leqslant c_i, \qquad \text{for all} \quad t \in [p, s_i],\ k \in N.$$

Furthermore if $s, t \in [p, s_i]$

$$| x_k(t) - x_k(s)| \leqslant \left| \int_s^t \| F_k(u, x_k(u))\| \, du \right| \leqslant (ac_i + b) \mid t - s \mid, \qquad k \in N,$$

and the sequence is also equicontinuous. By virtue of Ascoli–Arzelà theorem $\{x_k(t)\}$ contains a subsequence $\{x_{1k}(t)\}$ which converges uniformly on $[p, s_1]$. By the same argument this subsequence contains another subsequence $\{x_{2k}(t)\}$ which is uniformly convergent on $[p, s_2]$, and so on. Then the diagonal $\{z_k(t)\}$, where $z_k(t) = x_{kk}(t)$ converges uniformly on compacta to a function $z(t)$ which is defined on $[p, \infty)$, is absolutely continuous since

$$| z(t) - z(s)| \leqslant (ac_i + b) \mid t - s \mid, \qquad s, t \in [p, s_i]$$

and, for $t = p$, $z(p) \in \bar{B}(0, r)$. As all $z_k(t)$ with $k \geqslant h$, $h \in N$, satisfy (7_{hh}), by a well known result (see [10]), also $z(t)$ is solution of the same equation (7_{hh}). Thus $z(t)$ satisfies all equations (7_h) and, consequently, (6). But, as solution of (6) with data $(p, z(p)) \in Q$, $z(t) \to 0$ when $t \to \infty$, and there exists $T > 2p$ such that

$$| z(t)| < r/2, \qquad \text{for all} \quad t \in [T, \infty).$$

Take k so large to have definitively $s_{kk} > 2T$. Since $z_k(t) \to z(t)$ uniformly on $[T, 2T]$ there exists k such that

$$| z_k(t)| < r, \qquad \text{for all} \quad t \in [T, 2T].$$

This last inequality and (9) furnish the desired contradiction, being $2p < T < 2T < s_{kk}$. The proof of the theorem is complete.

The following two theorems are proved like in [11].

THEOREM 2. *Suppose that*:

(i) *$\{F_k\}$ is an infinite sequence of functions $F_k : R^+ \times E^n \to C^n$, $F_k \in \chi$, such that $F_k(t+p, x) = F_k(t, x)$, $p > 0$, $\|F_k(t, x)\| \leqslant a\,|x| + b$, $(a, b > 0)$ and, $F_1(t, x) \supseteq F_2(t, x) \supseteq \ldots$, $(t, x) \in R^+ \times E^n$;*

(ii) *every solution $x(t)$ of (6) with initial values $(t^0, x^0) \in [0, p] \times E^n$ approaches zero as $t \to \infty$.*

Then there exist $k \in N$ and $L \geqslant 1$ such that every solution $x(t)$ of (7_k) with initial values $(t^0, x^0) \in [0, p] \times E^n$ satisfies

$$|x(t)| \leqslant L\,|x^0|, \qquad \textit{for all} \quad t \in R^+. \tag{10}$$

Proof. Take $Q = [0, p] \times \bar{B}(0, 1)$. By Theorem 1, there exist $k \in N$ and $L \geqslant 1$ such that every solution $z(t)$ of (7_k) with data in Q satisfies

$$|z(t)| \leqslant L, \qquad \text{for all} \quad t \in R^+. \tag{11}$$

Let $x(t)$ be any other solution of (7_k) with data $(t^0, x^0) \in Q$. Let $x^0 \neq 0$ and set $z(t) = |x^0|^{-1} x(t)$. Dividing both sides of (7_k) by $|x^0|$ and taking into consideration the fact that F_k is homogeneous in x, we find that $z(t)$ is solution of the same equation (7_k). Since $(t^0, z^0) \in Q$, $z(t)$ satisfies (11) and so (10) remains proved for $x^0 \neq 0$. When $x^0 = 0$, since F_k is homogeneous in x, it is easily seen that the only solution of (7_k) with initial values $(t^0, 0)$ is $x(t) \equiv 0$. This completes the proof.

THEOREM 3. *Let $F \in \chi$, where F: $A = R^+ \times E^n \to C^n$. Suppose that there exist $\epsilon > 0$ and $L \geqslant 1$ such that every solution $z(t)$ of*

$$\dot{z} \in F(t, z) + \epsilon\bar{B}(0, |z|) \tag{12}$$

with initial values $(t^0, z^0) \in A$ satisfies

$$|z(t)| \leqslant L\,|z^0|, \qquad \textit{for all} \quad t \in [t^0, \infty). \tag{13}$$

Let $0 \leqslant \sigma < \epsilon$. Then every solution $x(t)$ of

$$\dot{x} \in F(t, x) + \sigma\bar{B}(0, |x|) \tag{14}$$

with initial values $(t^0, x^0) \in A$ satisfies

$$|x(t)| \leqslant L\,|x^0| \exp[-(\epsilon - \sigma)(t - t^0)], \qquad \textit{for all} \quad t \in [t^0, \infty).$$

Proof. Let $x(t)$ be any solution of (14) with $(t^0, x^0) \in A$. We want to show that $z(t) = x(t) \exp(\epsilon - \sigma)(t - t^0)$ is solution of (12). Indeed, from

$$\dot{z}(t) = [\dot{x}(t) + (\epsilon - \sigma)\,x(t)] \exp(\epsilon - \sigma)(t - t^0) \tag{16}$$

and

$$F(t, z(t)) + \epsilon\bar{B}(0, |z(t)|) = [F(t, x(t)) + \sigma\bar{B}(0, |x(t)|) + (\epsilon - \sigma)\,\bar{B}(0, |x(t)|)] \exp(\epsilon - \sigma)(t - t^0) \tag{17}$$

and the fact that $x(t)$ satisfies (14), it follows that the point at the second member of (16) belongs to the set at the second member of (17) for almost all $t \in [t^0, \infty)$. Thus $z(t)$ satisfies (12) and, a fortiori, (13), i.e., $|x(t)| \exp(\epsilon - \sigma)(t - t^0) \leqslant L |x^0|$, $t \in [t^0, \infty)$, which yields the inequality (15). The proof is complete.

4. Main Result

In this paragraph we shall state and prove the aforementioned generalization of Lyapunov's theorem. For this purpose we need the following

Lemma 3. *Let $f: R^+ \times E^n \to E^n$ be continuous, periodic in t of period $p > 0$ and Lipschitzian in $B = \{(t, y(t)): t \in R^+\}$ where $y(t) = y(t, 0, y^0)$ is a periodic solution of $\dot{x} = f(t, x)$ of period p. Let $D_f(t, y(t); z)$ be continuous as a function of $(t, z) \in R^+ \times E^n$. For each $t \in [0, p]$, denote by $\delta(t)$ the supremum of all numbers $0 < \delta \leqslant 1$ such that*

$$f(t, x) \in C(t, x - y(t)), \qquad \textit{for all} \quad x \in B(y(t), \delta)$$

where

$$C(t, x - y(t)) = f(t, y(t)) + D_f(t, y(t); x - y(t)) + \sigma B(0, |x - y(t)|)$$

and $\sigma > 0$. Then $\inf\{\delta(t): t \in [0, p]\} > 0$.

Proof. First we observe that, since $D_f(t, y(t); x - y(t)) + (\sigma/2)\bar{B}(0, |x - y(t)|)$ is an upper differential of f [3] at $(t, y(t))$, the set of which $\delta(t)$ is the supremum is nonvoid. To prove the statement of the lemma it is sufficient to show that $\delta(t)$ is lower semicontinuous on $[0, p]$. Suppose the contrary. There exist $\bar{t} \in [0, p]$, $\epsilon > 0$ and a monotone (say increasing) sequence $\{t_k\}$, $t_k \to \bar{t}$, such that

$$\delta(t_k) \leqslant \delta(\bar{t}) - \epsilon, \qquad \text{for all} \quad k \in N.$$

Let $k \in N$ be any. We have

$$f(t_k, x) \in C(t_k, x - y(t_k)), \qquad \text{for all} \quad x \in B(y(t_k), \delta(t_k)).$$

Let $\epsilon_1 > 0$ be such that $0 < \delta(t_k) \leqslant \delta(\bar{t}) - \epsilon < \delta(\bar{t}) - \epsilon_1 < \delta(\bar{t})$. By the definition of $\delta(t_k)$, there exists at least one point x_k satisfying $\delta(\bar{t}) - \epsilon < |x_k - y(t_k)| < \delta(\bar{t}) - \epsilon_1$ and

$$f(t_k, x_k) \notin C(t_k, x_k - y(t_k)), \qquad \text{for all} \quad k \in N. \tag{18}$$

We assume, without loss of generality, $x_k \to \bar{x}$. Then

$$f(\bar{t}, \bar{x}) \in C(\bar{t}, \bar{x} - y(\bar{t})) \qquad \text{since} \quad \bar{x} \in B(y(\bar{t}), \delta(\bar{t}))$$

where $C(\bar{t}, \bar{x} - y(\bar{t}))$ is open being $B(0, |\bar{x} - y(\bar{t})|)$ an open ball of positive radius. Thus, for some $\eta > 0$, $B(f(\bar{t}, \bar{x}), \eta) \subset C(\bar{t}, \bar{x} - y(\bar{t}))$. On the other hand, from the continuity of f and C, we obtain for a sufficiently large k

$$f(t_k, x_k) \in B(f(\bar{t}, \bar{x}), \eta/2) \subset C(t_k, x_k - y(t_k))$$

which contradicts (18). This completes the proof.

THEOREM 4. *Suppose that:*

(i) $y(t) = y(t, t^0, y^0)$, $(t^0, y^0) \in [0, p] \times E^n$, *is a periodic solution of*

$$\dot{x} = f(t, x) \tag{19}$$

where $f\colon A \to E^n$, $A = R^+ \times E^n$, *is continuous, periodic in t with period $p > 0$ and Lipschitzian in* $B = \{(t, y(t))\colon t \in R^+\}$;

(ii) *every solution* $x(t) = x(t, t^0, x^0)$, $(t^0, x^0) \in [0, p] \times E^n$, *of*

$$\dot{x} \in D_f(t, y(t); x) \tag{20}$$

where $D_f(t, y(t); x)$ *is continuous in A and such that* $\| D_f(t, y(t); x)\| \leqslant a\,|x| + b$, $(t, x) \in A$, $(a, b > 0)$, *satisfies* $x(t) \to 0$ *as* $t \to \infty$.

Then there exist constants $\delta > 0$, $L \geqslant 1$, $k \in N$ *such that every solution* $x(t)$ *of* (19), *defined for* $t \in [t^0, \infty)$ *and with initial values* $(t^0, x^0) \in [0, p] \times B(0, \delta)$, *satisfies*

$$|x(t) - y(t)| \leqslant L\,|x^0 - y^0|\exp[-c(t - t^0)], \quad c = \frac{1}{k(k+1)}, \quad t \in [t^0, \infty).$$

Proof. Under the stated hypotheses $D_f(t, y(t); z)$ exists and is periodic in t with period p. Set

$$F_k(t, y(t); z) = D_f(t, y(t); z) + (1/k)\,\bar{B}(0, |z|), \qquad k \in N.$$

F_k is an upper differential of f at $(t, y(t))$ [3]. Consider

$$\dot{z} \in F_k(t, y(t); z) + (1/k)\,\bar{B}(0, |z|), \qquad k \in N. \tag{21_k}$$

The hypotheses of Theorem 2 are fulfilled and there exist $k \in N$, $L \geqslant 1$ such that every solution $z(t)$ of (21_k) with initial values $(t^0, z^0) \in [0, p] \times E^n$ satisfies $|z(t)| \leqslant L\,|z^0|$, for all $t \in [t^0, \infty)$. By Theorem 3, taking $\epsilon = 1/k$ and $\sigma = 1/(k+1)$, every solution $z(t)$ of

$$\dot{z} \in F_k(t, y(t); z) + [1/(k+1)]\,\bar{B}(0, |z|) \tag{22}$$

with $(t^0, z^0) \in [0, p] \times E^n$ satisfies for $t \in [t^0, \infty)$

$$|z(t)| \leqslant L\,|z^0|\exp[-c(t - t^0)]. \tag{23}$$

Since $F_k(t, y(t); z)$ is an upper differential of f at $(t, y(t))$ there exist $\delta > 0$, which according to Lemma 3 can be choosen independent of t, such that

$$f(t, x) \in f(t, y(t)) + F_k(t, y(t); x - y(t)) \quad \text{for all } x \in B(y(t), \delta),\ t \in R^+. \quad (24)$$

Let $x(t)$ be any other solution of (19) defined for $t \in [t^0, \infty)$ with initial data

$$(t^0, x^0) \in [0, p] \times B(y^0, \delta_1) \qquad \text{where} \quad 0 < \delta_1 < \delta/2L.$$

We claim that $z(t) = x(t) - y(t)$ satisfies

$$\dot{z} \in F_k(t, y(t); z) \tag{25}$$

on some interval $[t^0, t^0 + s)$, for a sufficiently small $s > 0$. Indeed, from the identity

$$\dot{x}(t) - \dot{y}(t) = f(t, x(t)) - f(t, y(t))$$

and (24), we obtain

$$\dot{x}(t) - \dot{y}(t) \in f(t, y(t)) + F_k(t, y(t); x(t) - y(t)) - f(t, y(t))$$

and (25) is satisfied providing $x(t) \in B(y(t), \delta)$, say for $t \in [t^0, t^0 + s)$, $s > 0$. Such an s certainly exists because $x^0 \in B(y^0, \delta_1)$, $\delta_1 < \delta/2L$, and $x(t)$, $y(t)$ are continuous.

Next we show that $z(t)$ satisfies (25) for all $t \in [t^0, \infty)$. To this end define

$$\bar{t} = \sup\{t^0 + s\colon z(t) \text{ satisfies (25) on } [t^0, t^0 + s),\ s > 0\}$$

and, in view of an indirect proof, suppose $\bar{t} < \infty$. On $[t^0, \bar{t})$, $z(t)$ satisfies (25) hence a fortiori (22). Then, from (23),

$$z(t) \in B(0, L \mid z^0 \mid) \subset B(0, \delta/2), \qquad \text{for all} \quad t \in [t^0, \bar{t}),$$

being $\mid z^0 \mid = \mid x^0 - y^0 \mid < \delta_1 < \delta/2L$. Thus $z(\bar{t}) \in \bar{B}(0, \delta/2)$ i.e.

$$x(\bar{t}) \in \bar{B}(y(\bar{t}), \delta/2). \tag{26}$$

But now $z(t)$ can be continued beyond $\bar{t}$ as a solution of (25) since, by a continuity argument, from (24) and (26) we get on some small time interval $[\bar{t}, \bar{t} + s)$, $s > 0$,

$$f(t, x(t)) \in f(t, y(t)) + F_k(t, y(t); x(t) - y(t))$$

that is

$$\dot{x}(t) - \dot{y}(t) \in F_k(t, y(t); x(t) - y(t))$$

and $z(t)$ remains solution of (25) also on $[\bar{t}, \bar{t} + s)$, a contradiction to the definition of $\bar{t}$. This proves that $z(t)$ is a solution of (25) (hence in particular of (22)) which is defined all over $[t^0, \infty)$. The estimate (23) holds and the proof is complete.

EXAMPLE 2. The equation $\dot{x} = f(t, x) - \sin t$, where

$$f(t, x) \begin{cases} = -(x - \cos t) \left[\sin \dfrac{1}{\cos t - x} + 2\right] & \text{if } \ x \neq \cos t, \\ = 0 & \text{if } \ x = \cos t, \end{cases}$$

has the periodic solution $x = \cos t$. The corresponding variational equation, along $x = \cos t$, is multivalued, that is

$$\dot{z} \in D_f(t, \cos t; z)$$

where

$$D_f(t, \cos t; z) \begin{cases} = [-3z, -z] & \text{if } \ z \geqslant 0, \\ = [-z, -3z] & \text{if } \ z \leqslant 0. \end{cases}$$

It is easy to see that the hypotheses of Theorem 4 are satisfied: consequently all solutions of the given equation starting sufficiently near to $x = \cos t$ tend to this solution as $t \to \infty$.

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